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Existence of a Solution of a Boundary Value Problem for a Class of Discontinuous Nonlinear Systems

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1. INTRODUCTION

In this paper we investigate periodic solutions and related boundary value problems for a family of nonlinear differential equations in which the nonlinearity may be discontinuous (and/or unbounded). Specifically, we establish the existence of harmonic solutions for equations of the type

$$\frac{d^2x}{dt^2} + x + f(x) = a \cos \omega t, \qquad (DE, f)$$

where the function f is odd, satisfies $xf(x) \geqslant 0$ for all x and has a jump discontinuity. All quantities are real-valued, and the constants a and ω are assumed always to satisfy $a < 0$ and $\omega > 1$.

What we ordinarily mean by a harmonic (periodic) solution of (DE, f) is a solution with period $2\pi/\omega$ which is exactly out of phase (in phase when $0 < \omega < 1$) with the forcing function $a \cos \omega t$. Now it is easily verified (see [5]) that a solution of the boundary value problem consisting of (DE, f) and the boundary conditions

$$\dot{x}(0) = 0, \qquad x(\tau) = 0, \tag{1}$$

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(where $\tau = \pi/2\omega$), when extended for all t, yields such a harmonic (periodic) solution of (DE, f). Because it is the boundary value problem, denoted henceforth by $BVP(f)$, which is treated directly below, and because of the relationship between $BVP(f)$ and the periodicity problem, *in this paper by a harmonic solution of* (DE, f) *we shall mean a solution of* $BVP(f)$ *which is positive for* $0 \leqslant t < \tau$.

When f is uniformly bounded and smooth (except perhaps for a jump at $x = 0$), (DE, f) is known [3] to possess a harmonic solution. The restriction that f experience a jump only at the origin is crucial in [3]; in this case a new (unsymmetric) smooth nonlinearity is constructed which coincides with f for $x > 0$. This technique does not work, however, when f has a jump discontinuity at points other than at $x = 0$. Our principal purpose here is to consider this problem. Further, it is shown that the restriction that f be uniformly bounded may be relaxed considerably: in Section 4, we consider a class of unbounded nonlinearities which satisfy suitable growth conditions.

The existence of a solution of $BVP(f)$ is established below by use of the well-known point-mapping method; to employ this method we need to know that, for each value of the real parameter α in a certain interval, the initial value problem (DE, f), $x(0) = \alpha$, $\dot{x}(0) = 0$ possesses a unique solution on $[0, \tau]$ which depends continuously on α. This initial value problem is treated in Section 5.

For ease of exposition we have restricted ourselves to functions f with a single discontinuity. Similar results can be obtained for a broader class of problems, involving nonlinearities with multiple discontinuities. Such problems will be discussed elsewhere. A particular case of multiple discontinuities, in which the nonlinearity is piecewise constant, is treated in [5] by a special technique.

2. Preliminaries

Let $b = a/(1 - \omega^2)$.

F will denote the class of functions f with the following properties: f is odd, $f(x) \geqslant 0$ when $x > 0$, $f(x) \not\equiv 0$, and f is continuous for $x \geqslant 0$ except perhaps at $x = \sigma$ $(0 < \sigma < b)$, where it has a jump discontinuity.

In Sections 3, 4 and 5 we introduce subsets of F denoted, respectively, by F_M (f uniformly bounded by a positive constant M), F_g (f unbounded but subject to a growth restriction), and F_L ($f \in F_M \cup F_g$ but in addition satisfying a Lipschitz condition in each interval of continuity).

When f is piecewise continuous we mean by a solution of (DE, f) on an interval I a function $x(t)$ which is continuously differentiable on this interval, possesses a piecewise continuous second derivative and satisfies (DE, f)

wherever the second derivative exists. In case f is continuous a solution is to have a continuous second derivative and satisfy (DE, f) at all points of I.

Let $IVP(\alpha, f)$ denote the initial value problem of solving (DE, f) subject to the initial conditions

$$x(0) = \alpha, \qquad \dot{x}(0) = 0. \tag{2}$$

By use of the method of variation of parameters, a solution of $IVP(\alpha, f)$ on an interval I containing the origin is easily seen to satisfy

$$x(t) = (\alpha - b) \cos t + b \cos \omega t - \int_0^t \sin(t - s) f(x(s))\, ds, \qquad (t \in I). \tag{3}$$

Conversely, a continuous solution of (3) on I is a solution of $IVP(\alpha, f)$ on I.

Let $x(t; \alpha, f)$ denote a solution of $IVP(\alpha, f)$ which is known to exist and be unique on $[0, \tau]$. (When f is Lipschitzian, such a unique solution is known to exist (for instance, see [4]); this is less evident, however, when f is piecewise continuous. We establish in Section 5 the existence, uniqueness and continuous dependence on initial data of solutions of $IVP(\alpha, f)$ when f belongs to a subset of our class F of piecewise continuous nonlinearities.)

Let A be a set of values of α for which (unique) $x(t; \alpha, f)$ exists. We define on A a mapping T by the formula

$$T(\alpha) = x(\tau; \alpha, f).$$

Clearly $x(t; \alpha, f)$ will be a harmonic solution of (DE, f) for those values of α for which $T(\alpha) = 0$ and $x(t; \alpha, f) > 0$ for $0 \leqslant t < \tau$. Any such value of α will henceforth be called a *harmonic point.*

The following theorem is a restatement, under weaker assumptions, of one appearing originally in [2] and [3].

COMPARISON THEOREM. *Suppose that f_1 and f_2 are piecewise continuous functions and that $f_1(r) \geqslant f_2(s)$ for all $r > 0$, $s > 0$. Let α_1 be a harmonic point of (DE, f_1) and α_2 a harmonic point of (DE, f_2). Then*

$$\alpha_1 \geqslant \alpha_2 .$$

Proof. Let $x_i(t; a_i, f_i)$ be a harmonic solution of (DE, f_i) $(i = 1, 2)$. Then, using (3), the harmonic points α_i $(i = 1, 2)$ are given by

$$\alpha_i = b + \sec \tau \int_0^\tau \sin(\tau - s) f_i(x_i(s))\, ds,$$

so that

$$\alpha_1 - \alpha_2 = \sec \tau \int_0^\tau \sin(\tau - s) \{f_1(x_1) - f_2(x_2)\}\, ds \geqslant 0.$$

Here we have used the facts that $0 \leqslant \tau - s \leqslant \pi/2$ and $x_i(s; \alpha_i, f_i) > 0$ $(i = 1, 2)$ for $s \in (0, \tau)$.

In the linear case, $f(x) \equiv 0$, the equation $(DE, 0)$ has a unique harmonic point $b = a/(1 - \omega^2)$. The ideal relay equation

$$\frac{d^2x}{dt^2} + x + M \operatorname{sgn} x = a \cos \omega t, \qquad (DE, M \operatorname{sgn})$$

where M is a positive constant and $\operatorname{sgn} x = \operatorname{signum} x = x/|x|$ $(x \neq 0)$, is known [7] to have a harmonic point, which we denote by d_M:

$$d_M = b + M(\sec \tau - 1). \qquad (4)$$

According to the Comparison Theorem, d_M is the unique harmonic point of $(DE, M \operatorname{sgn})$.

3. Existence of a Harmonic Solution: f Uniformly Bounded

In this section we assume always that f is contained in F_M, the subset of F consisting of functions which satisfy $|f(x)| \leqslant M$ for all x, where M is a given positive constant.

When $f \in F_M$ it is not hard to see (since $f(x) \not\equiv 0$) that there is a subinterval of $(0, d_M)$, on which $f(x) > 0$, and there is also a subinterval of $(0, d_M)$ on which $f(x) < M$. These facts are needed in Lemma 2 and Corollary 1 of Lemma 3.

From the Comparison Theorem follows immediately

Lemma 1. *If α is a harmonic point of (DE, f),*

$$b \leqslant \alpha \leqslant d_M.$$

Thus, for $f \in F_M$ we know that we must look for a harmonic point (if any exists) in the interval $[b, d_M]$. To show that one does exist (that is, that there exists at least one harmonic solution) we use the well-known point-mapping method [1]; namely, we demonstrate that T is a continuous mapping and carries the interval $[b, d_M]$ into an interval which contains zero in its interior (so that at least one point in $[b, d_M]$ is mapped into zero under T).

To this end we show below that $T(d_M) > 0$, $T(b) < 0$ and $\dot{x}(t; \alpha, f) < 0$ on $0 < t \leqslant \tau$ for any $\alpha \in [b, d_M]$. The continuity of $T(\alpha)$ in α follows from the continuous dependence of $x(t; \alpha, f)$ on the initial datum α, which is established in Section 5. These facts yield the desired result, Theorem 1.

In Lemma 2 and Lemma 3 we assume that for each $\alpha \in [b, d_M]$ there exists a solution of $IVP(\alpha, f)$ which is defined for all $t \in [0, \tau]$. In Section 5 we demonstrate that this is not an empty assumption.

LEMMA 2. $T(d_M) = x(\tau; d_M, f) > 0$.

Proof. Since $|f(x)| \leqslant M$,

$$\int_0^\tau \sin(\tau - s) f(x(s))\, ds \leqslant M \int_0^\tau \sin(\tau - s)\, ds$$
$$= M(1 - \cos \tau).$$

But

$$\begin{aligned} x(\tau; d_M, f) &= (d_M - b) \cos \tau - \int_0^\tau \sin(\tau - s) f(x(s))\, ds \\ &\geqslant (d_M - b) \cos \tau - M(1 - \cos \tau) \\ &= [d_M - b - M(\sec \tau - 1)] \cos \tau \\ &= 0. \end{aligned}$$

Thus $x(\tau; d_M, f) \geqslant 0$. Now suppose $x(\tau; d_M, f) = 0$. Then $x(t; d_M, f)$ takes all values in $0 \leqslant x \leqslant d_M$, on a subinterval of which $f(x) < M$. Therefore, retracing the steps above,

$$\int_0^\tau \sin(\tau - s) f(x(s))\, ds < M(1 - \cos \tau)$$

and

$$x(\tau; d_M, f) > 0.$$

COROLLARY. *If α is a harmonic point of (DE, f) then $\alpha < d_M$.*

Proof. From Lemma 1, $\alpha \leqslant d_M$. But if $\alpha = d_M$, $x(\tau; d_M, f) = 0$, contradicting Lemma 2.

LEMMA 3. *Suppose $b\omega \geqslant M$. Then for $\alpha \in [b, d_M]$, $\dot{x}(t; \alpha, f) < 0$ on $0 < t \leqslant \tau$.*

Proof.

$$\begin{aligned} \dot{x}(t; \alpha, f) &= (b - \alpha) \sin t - b\omega \sin \omega t - \int_0^t \cos(t - s) f(x(s))\, ds \\ &\leqslant (b - \alpha) \sin t - b\omega \sin \omega t + M \int_0^t \cos(t - s)\, ds \\ &\leqslant (b - \alpha) \sin t - b\omega(\sin \omega t - \sin t) \\ &\leqslant -\eta(t) < 0 \end{aligned}$$

where

$$\eta(t) = b\omega(\sin \omega t - \sin t).$$

COROLLARY 1. $T(b) = x(\tau; b, f) < 0$.

Proof. On the contrary, assume that $x(\tau; b, f) \geqslant 0$. Then, since $\dot{x}(t; b, f) < 0$, $x(t; b, f) > 0$ for $0 \leqslant t < \tau$. This implies $f(x(t)) > 0$, so that from (3)

$$x(\tau; b, f) = -\int_0^\tau \sin(\tau - s) f(x(s))\, ds < 0.$$

This contradiction completes the proof.

COROLLARY 2. *If α is a harmonic point of (DE, f), then $\alpha > b$.*

On the basis of the preceding lemmas, which establish the fact that the mapping T carries $[b, d_M]$ into an interval containing zero in its interior, and in view of continuity of the solution $x(t; \alpha, f)$ in α (which will be demonstrated in Section 5), we have proved

THEOREM 1. *Suppose $f \in F_M$. Then there exists an α, $\alpha \in (b, d_M)$, which is a harmonic point of (DE, f), provided*

$$b\omega \geqslant M.$$

4. EXISTENCE OF A HARMONIC SOLUTION: f UNBOUNDED

Now we extend the results of the preceding section to a class of unbounded nonlinearities. In this section it is assumed always that f belongs to F_g, the subset of F consisting of functions f which obey the following growth condition: there exists a positive number $\bar{x}$ (where $\bar{x} > b > \sigma > 0$) such that for $x > 0$

$$\begin{aligned} f(x) &\leqslant f(\bar{x}) = \frac{\bar{x} - b}{\sec \tau - 1}, \qquad (x \leqslant \bar{x}), \\ f(x) &< \frac{x - b}{\sec \tau - 1}, \qquad (x > \bar{x}). \end{aligned} \tag{5}$$

To motivate this extension, let us consider an arbitrary $f \in F$. For a given $\alpha \geqslant b$, suppose $x(t; \alpha, f)$ is a harmonic solution of (DE, f). Since $f(x(t)) \geqslant 0$ for $0 \leqslant t \leqslant \tau$, it follows from $\dot{x}(t; \alpha, f) < 0$ on $(0, \tau]$ (modifying slightly the proof of Lemma 3) that $x(t; \alpha, f) < \alpha$ on $(0, \tau)$. Define the odd truncated function f_0 by

$$f_0(x) = \begin{cases} f(x), & 0 \leqslant x \leqslant \alpha \\ f(\alpha), & x > \alpha. \end{cases}$$

Since $x(t; \alpha, f)$ takes values only in $-\alpha \leqslant x \leqslant \alpha$, where f and f_0 are identical, $x(t; \alpha, f)$ is a harmonic solution of

$$\frac{d^2x}{dt^2} + x + f_0(x) = a \cos \omega t \qquad (DE, f_0)$$

as well as of (DE, f). Since $|f_0(x)| \leqslant f(\alpha)$, we find, from the Comparison Theorem of Section 2, using $M = f(\alpha)$, that

$$b \leqslant \alpha \leqslant b + f(\alpha)\,[\sec \tau - 1]. \tag{6}$$

Inequality (6) imposes a necessary condition to be satisfied by the amplitude α of a harmonic solution (if any exists) of (DE, f). Thus, if for all x greater than some $\bar{x} > 0$, $x > b + f(x)\,[\sec \tau - 1]$, there cannot exist a harmonic solution with amplitude greater than $\bar{x}$. This motivates the definition of the set F_g.

On the basis of the preceding discussion, we have

LEMMA 6. *Suppose $f \in F_g$. If α is a harmonic point of (DE, f), then*

$$b \leqslant \alpha \leqslant \bar{x}. \tag{7}$$

Thus, in order to investigate harmonic solutions of (DE, f), for $f \in F_g$, it is sufficient to consider solutions of $IVP(\alpha, f)$ for $b \leqslant \alpha \leqslant \bar{x}$.

Define the truncated function $\bar{f}$ by

$$\bar{f}(x) = \begin{cases} f(x), & 0 \leqslant x \leqslant \bar{x} \\ \dfrac{\bar{x} - b}{\sec \tau - 1}, & x > \bar{x}. \end{cases} \tag{8}$$

Clearly $|\bar{f}(x)| \leqslant (\bar{x} - b)/(\sec \tau - 1)$, and Lemmas 2 and 3 can be restated in terms of $\bar{f}$, and thus in terms of f. We have, for instance,

LEMMA 2′. *Let $x(t; \bar{x}, f)$ be a solution of $IVP(\bar{x}; f)$. Then $x(\tau; \bar{x}, f) > 0$. If α is a harmonic point of (DE, f), then $\alpha < \bar{x}$.*

We shall merely state the existence result.

THEOREM 2. *Let $f \in F_g$. Then there exists an $\alpha \in (b, \bar{x})$ which is a harmonic point of (DE, f) provided*

$$b\omega \geqslant \frac{\bar{x} - b}{\sec \tau - 1}.$$

5. Existence, Uniqueness and Continuous Dependence of a Solution of $IVP(\alpha, f)$

Let F_L be the subset of F consisting of those elements in F_M or F_g which, except at $x = \pm\sigma$, are Lipschitzian: for $x \geqslant 0$

$$f(x) = \begin{cases} f_1(x), & x \leqslant \sigma \\ f_2(x), & x \geqslant \sigma \end{cases}$$

where f_1 and f_2 are Lipschitz continuous on their respective domains of definition. Obviously $F_L \subset (F_M \cup F_g)$.

Throughout this section it is assumed that $\alpha \in [b, \bar{x}]$, $\bar{x} > b > \sigma > 0$, and that $f \in F_L$. Calculations and results are stated explicitly for members f of F_g (which satisfy the growth restriction (5)). They are easily modified for members f of F_M (uniformly bounded).

We work with the trajectories of solutions in t, x, $\dot{x}$-space (phase space). Given the constant $N > 0$ (which for the moment is arbitrary), let

$$D = D_1 \cup D_2 \cup D_3$$

where

$$D_1 = \{(t, x, \dot{x}) \mid 0 \leqslant t \leqslant \tau,\ \sigma \leqslant x \leqslant \bar{x},\ |\dot{x}| \leqslant N\}$$
$$D_2 = \{(t, x, \dot{x}) \mid 0 \leqslant t \leqslant \tau,\ -\sigma \leqslant x \leqslant \sigma,\ |\dot{x}| \leqslant N\}$$
$$D_3 = \{(t, x, \dot{x}) \mid 0 \leqslant t \leqslant \tau,\ -\bar{x} \leqslant x \leqslant -\sigma,\ |\dot{x}| \leqslant N\}.$$

We shall prove that for each $\alpha \in [b, \bar{x}]$ the solution of $IVP(\alpha, f)$ can be continued uniquely up to $t = \tau$ and that this solution depends continuously on the initial datum α.

Since in D_1 f is Lipschitz continuous, there exists a t_1 ($0 < t_1 \leqslant \tau$) such that for $0 < t < t_1$, $IVP(\alpha, f)$ has a unique solution whose trajectory is in D_1. This trajectory approaches the boundary of D_1 as $t \to t_1$ (for example, see [8]). Let us denote this solution by $x(t; \alpha, f)$. Then $x(t; \alpha, f)$ satisfies the integral equation (3) (with f replaced by f_2) on $[0, t_1)$. For $0 < t < t_1$

$$\begin{aligned} \dot{x}(t; \alpha, f) &= -(\alpha - b)\sin t - b\omega \sin \omega t - \int_0^t \cos(t - s) f_2(x(s))\, ds \\ &\leqslant -(\alpha - b)\sin t - b\omega \sin \omega t + \frac{\bar{x} - b}{\sec \tau - 1} \sin t \\ &\leqslant -\eta(t) < 0, \end{aligned}$$

where $\eta(t) = b\omega(\sin \omega t - \sin t)$, provided

$$b\omega \geqslant \frac{\bar{x} - b}{\sec \tau - 1}. \tag{9}$$

(For the remainder of this section we assume that (9) holds.) Further,

$$
\begin{aligned}
-\dot{x}(t;\alpha,f) &\leqslant (\alpha-b)\sin t + b\omega\sin\omega t + \frac{\bar{x}-b}{\sec\tau-1}\int_0^t \cos(t-s)\,ds \\
&\leqslant (\bar{x}-b)\sin\tau + b\omega + \frac{\bar{x}-b}{1-\cos\tau}(\sin\tau\cos\tau) \\
&= b\omega + \frac{\bar{x}-b}{1-\cos\tau}(\sin\tau).
\end{aligned}
$$

Therefore, if we choose

$$N > b\omega + \frac{\bar{x}-b}{1-\cos\tau}(\sin\tau),$$

$$0 > \dot{x}(t;\alpha,f) > -N.$$

Thus, the trajectory of solution $x(t;\alpha,f)$, for $0 \leqslant t < t_1$, is bounded away from the boundaries $\dot{x} = \pm N$ and remains below the plane $x = \bar{x}$. Hence, as $t \to t_1$ the trajectory either approaches the boundary $t = \tau$ (in which case $t_1 = \tau$) or the boundary $x = \sigma$. Now if $t_1 = \tau$ we have a well defined solution of (DE, f) on $0 \leqslant t \leqslant \tau$ which depends continuously on α. When $t_1 < \tau$ define

$$v_1(t_1, \alpha) = \lim_{t\to t_1-0} \dot{x}(t;\alpha,f) < 0.$$

Thus $x(t;\alpha,f)$ is not tangent to the plane $x = \sigma$ at $t = t_1$. Clearly v_1 is a continuous function of (t_1, α). Now t_1 is obtained by solving the functional equation

$$x(t_1;\alpha,f) = \sigma.$$

Since $x(t;\alpha,f)$ is a strictly decreasing function of t, there exists a unique solution $t_1 = u_1(\alpha)$, where $u_1(\alpha)$ is a continuous function of α.

We now continue this solution for $t > t_1$ as the unique solution of the initial value problem

$$\frac{d^2x}{dt^2} + x + f_1(x) = a\cos\omega t$$

$$x(t_1) = \sigma, \qquad \dot{x}(t_1) = v_1 < 0$$

in D_2. Since f_1 is Lipschitz continuous in D_2, there exists a $t_2 (t_1 < t_2 \leqslant \tau)$ such that the unique continuation remains in D_2 for $t_1 < t < t_2 \leqslant \tau$ and depends continuously on the "initial" data t_1, v_1. Since these "initial" data themselves are continuous functions of α, we are assured of continuous dependence on α of the extended solution on $0 \leqslant t < t_2 \leqslant \tau$.

In order to gain insight into subsequent continuation of this solution, let us consider the comparison problem

$$\frac{d^2x}{dt^2} + x + \frac{\bar{x} - b}{\sec \tau - 1} = a \cos \omega t,$$

$x(0) = b$, $\dot{x}(0) = 0$. The unique solution of this problem is given by

$$x_\ell(t) = \frac{\bar{x} - b}{\sec \tau - 1} (\cos t - 1) + b \cos \omega t.$$

On their common interval of existence, $x_\ell(t)$ provides a lower bound for $x(t; \alpha, f)$:

$$\begin{aligned}
x(t; \alpha, f) - x_\ell(t) &= (\alpha - b) \cos t - \frac{\bar{x} - b}{\sec \tau - 1} (\cos t - 1) \\
&\quad - \int_0^t \sin(t - s) f(x(s))\, ds \\
&\geqslant (\alpha - b) \cos t - \frac{\bar{x} - b}{\sec \tau - 1} (\cos t - 1) \\
&\quad + \frac{\bar{x} - b}{\sec \tau - 1} (\cos t - 1) \\
&= (\alpha - b) \cos t \\
&\geqslant 0.
\end{aligned}$$

Since $x_\ell(t)$ is a strictly decreasing function of t on $[0, \tau]$, $x(t; \alpha, f)$ remains above the boundary $x = -\sigma$ for $0 \leqslant t \leqslant \tau$ (and therefore $t_2 = \tau$) if

$$x_\ell(\tau) = \frac{\bar{x} - b}{\sec \tau - 1} (\cos \tau - 1) = -(\bar{x} - b) \cos \tau > -\sigma$$

or

$$\sigma > (\bar{x} - b) \cos \tau. \tag{10}$$

Remark. If we assume that

$$b > \sigma > b\omega(1 - \cos \tau)$$

then (9) implies (10); for from (9)

$$\bar{x} - b \leqslant b\omega(\sec \tau - 1) < \sigma \sec \tau.$$

If (10) does not hold, the trajectory for $x(t; \alpha, f)$ may approach the boundary $x = -\sigma$ as $t \to t_2$. It is easily verified that

$$v_2(t_2, \alpha) = \lim_{t \to t_2 - 0} \dot{x}(t; \alpha, f) < 0.$$

We now continue the solution for $t > t_2$ as the unique solution of the initial value problem

$$\frac{d^2x}{dt^2} + x + f_2(x) = a \cos \omega t$$

satisfying $x(t_2) = -\sigma$, $\dot{x}(t_2) = v_2 < 0$. The trajectory thus obtained remains above the plane $x = -\bar{x}$ (and therefore reaches the plane $t = \tau$); for, from the comparison problem,

$$x(t; \alpha, f) \geqslant x_\ell(\tau) = -(\bar{x} - b) \cos \tau > -\bar{x}.$$

The above discussion is summarized in

THEOREM 3. *Suppose $f \in F_L \cap F_g$, where $0 < \sigma < b < \bar{x}$. Then for each $\alpha \in [b, \bar{x}]$ there exists a unique solution $x(t; \alpha, f)$ of the initial value problem $IVP(\alpha, f)$ on $0 \leqslant t \leqslant \tau$ which depends continuously on α, provided*

$$b\omega \geqslant \frac{\bar{x} - b}{\sec \tau - 1}.$$

In phase space the trajectory of this solution crosses each plane of discontinuity $x = \pm \sigma$ at most once.

If further $\sigma > b\omega(1 - \cos \tau)$ the unique solution of $IVP(\alpha, f)$ remains above the plane $x = -\sigma$ for $0 \leqslant t \leqslant \tau$; that is, the trajectory intersects at most one plane of discontinuity of f. (If $f \in F_L \cap F_M$, replace $\bar{x}$ above by d_M.)

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